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THE ZEROES OF NONNEGATIVE CURVATURE OPERATORS

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The Riemannian sectional curvature of a Riemannian manifold is a real-valued function σ on the Grassmann bundle of tangent 2-planes of M. Although there exists a large body of theorems relating the curvature of M to various topological and geometric properties of M, relatively little is known of a general nature about the behavior of the function σ itself. In particular, the critical point behavior of σ has been analyzed only in very special cases [3], [4]. In this paper we consider the pointwise behavior of σ; that is, we consider the restriction of σ to the Grassmann manifold of tangent 2-planes at a point $m \in M$. We are then able to describe completely the structure of the sets of points in this manifold where σ assumes its minimum and maximum. In particular, for spaces of nonnegative curvature we describe the set of points where σ assumes the value zero.

To be more specific, let G denote the Grassmann manifold of oriented tangent 2-planes at m. G is in a natural way a submanifold of the vector space Λ^2 of 2-vectors at m. Since G is a 2-fold covering space of the manifold of (unoriented) 2-planes at m, we may regard σ as a function on G. We shall show that the minimum and maximum sets of σ are intersections with G of linear subspaces of Λ^2. Moreover every such intersection can occur, for example as the minimum set of some curvature function σ on G.

The case of nonnegative curvature $\sigma \geq 0$ will occupy most of our attention here. One reason for this is that the general result on the minimum set of σ is an elementary consequence of the result for $\sigma \geq 0$, and another is that this case is the one most likely to yield applications. For example, it follows from our description of the minimum set that if $\sigma \geq 0$ and relative to some coordinate system the "diagonal" curvature components R_{ijij} are all zero at m, then in fact the curvature tensor R is zero at m.

Given a space M of nonnegative curvature and given $m \in M$, the linear subspace of Λ^2 whose intersection with G is the zero set of σ is obtained as follows. The curvature tensor R of M at m can be regarded as a self-adjoint linear operator on Λ^2. Letting $\mathscr{R}$ denote the vector space of all self-adjoint linear operators ("curvature operators") on Λ^2, the subset $\mathscr{B}$ consisting of those

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which come from Riemannian structures (i.e., those satisfying the first Bianchi identity) is a linear subspace of $\mathscr{R}$. The orthogonal complement $\mathscr{S}$ of $\mathscr{B}$ in $\mathscr{R}$ is the set of all curvature operators whose associated Riemannian curvature function is identically zero. Our theorem asserts that there exists an operator $S \in \mathscr{S}$ such that the zero set of σ (also called the zero set of R) is precisely $G \cap \operatorname{Ker}(R - S)$.

The idea of the proof is first to show that for each P in the zero set there exists an $S \in \mathscr{S}$ such that $P \in \operatorname{Ker}(R - S)$, second to observe that there is a unique such S orthogonal to the subspace of $\mathscr{S}$ annihilating P, and finally to piece these unique operators together to build one which works simultaneously for all P in the zero set.

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1. $\mathscr{S}$ and the Grassmann quadratic 2-relations

We begin by analyzing the space $\mathscr{S}$ complementary in $\mathscr{R}$ to the subspace $\{R \in \mathscr{R} \mid R$ satisfies the Bianchi identity$\}$. We shall exhibit a natural isomorphism between $\mathscr{S}$ and Λ^4 and shall establish the relationship between $\mathscr{S}$ and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in Λ^2.

Let V be an n-dimensional real vector space with inner product $\langle\ ,\ \rangle$ (e.g., $V =$ the tangent space at a point of a Riemannian manifold). For k an integer ≥ 0, let $\Lambda^k = \Lambda^k(V)$ denote the space of k-vectors of V, equipped with inner product given by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det[\langle u_i, v_j \rangle] , \qquad u_i, v_i \in V .$$

Let G denote the Grassmann manifold of oriented 2-dimensional subspaces of V; we identify G with the submanifold of Λ^2 consisting of decomposable 2-vectors of length 1 by $P \leftrightarrow u \wedge v$ where $\{u, v\}$ is any oriented orthonormal basis for P. Let $\mathscr{R}$ denote the space of self-adjoint linear operators on Λ^2, equipped with inner product given by $\langle R, S \rangle = \operatorname{trace} R \circ S$, $R, S \in \mathscr{R}$. Elements of $\mathscr{R}$ will be called curvature operators on V. Given $R \in \mathscr{R}$, its sectional curvature is the function $\sigma_R \colon G \to \boldsymbol{R}$ defined by $\sigma_R(P) = \langle RP, P \rangle$, $P \in G$. Each $R \in \mathscr{R}$ can be naturally identified with a 2-form on V with values in the vector space of skew-symmetric endomorphisms of V by

$$\langle R(u, v)(w), x \rangle = R(u \wedge v, w \wedge x) , \qquad u, v, w, x \in V .$$

We can then consider the subspace $\mathscr{B}$ of $\mathscr{R}$ consisting of those $R \in \mathscr{R}$ which satisfy the first Bianchi identity: $R \in \mathscr{B}$ if and only if

$$R(u, v)w + R(v, w)u + R(w, u)v = 0$$

for all $u, v, w \in V$. Set $\mathscr{S} = \mathscr{B}^{\perp}$, the orthogonal complement of $\mathscr{B}$ in $\mathscr{R}$.

We construct, for each $\xi \in \Lambda^4$, an operator $S_\xi \in \mathscr{S}$ as follows. Given ξ, define $S_\xi \colon \Lambda^2 \to \Lambda^2$ by

$$\langle S_\xi(\alpha), \beta\rangle = \langle \alpha \wedge \beta, \xi\rangle\,, \qquad \alpha, \beta \in \Lambda^2\,.$$

Clearly $S_\xi \in \mathscr{R}$. To see that $S_\xi \in \mathscr{S}$ we need the following

Lemma 1.1. *Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V. For $1 \leq i, j, k, l \leq n$, set $S_{ijkl} = S_{e_i \wedge e_j \wedge e_k \wedge e_l}$. Then, for $R \in \mathscr{R}$,*

$$\langle R, S_{ijkl}\rangle = 2[\langle R(e_i \wedge e_j), e_k \wedge e_l\rangle + \langle R(e_j \wedge e_k), e_i \wedge e_l\rangle + \langle R(e_k \wedge e_i), e_j \wedge e_l\rangle]\,.$$

Proof.

$$\begin{aligned}\langle R, S_{ijkl}\rangle &= \operatorname{tr} R \circ S_{ijkl} = \textstyle\sum_{\alpha<\beta} \langle R \circ S_{ijkl}(e_\alpha \wedge e_\beta), e_\alpha \wedge e_\beta\rangle \\ &= \textstyle\sum_{\alpha<\beta} \langle S_{ijkl}(e_\alpha \wedge e_\beta), R(e_\alpha \wedge e_\beta)\rangle \\ &= \textstyle\sum_{\alpha<\beta} \langle S_{ijkl}(e_\alpha \wedge e_\beta), \sum_{\gamma<\delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta\rangle e_\gamma \wedge e_\delta\rangle \\ &= \textstyle\sum_{\alpha<\beta} \sum_{\gamma<\delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta\rangle \\ &\qquad \times \langle e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta, e_i \wedge e_j \wedge e_k \wedge e_l\rangle\,.\end{aligned}$$

Collecting terms completes the proof.

Proposition 1.2. *$\xi \mapsto S_\xi$ maps Λ^4 isomorphically onto $\mathscr{S}$. Moreover $\xi \mapsto (1/\sqrt{6})S_\xi$ is an isometry.*

Proof. Clearly $\xi \mapsto S_\xi$ is a linear map from Λ^4 into $\mathscr{R}$. Since $\{e_i \wedge e_j \wedge e_k \wedge e_l \mid i < j < k < l\}$ is an (orthonormal) basis for Λ^4, and the images S_{ijkl} of the basis vectors are all in $\mathscr{S}$ ($\langle R, S_{ijkl}\rangle = 0$ for all $R \in \mathscr{B}$ by Lemma 1.1), it follows that $\xi \mapsto S_\xi$ maps Λ^4 into $\mathscr{S}$. In fact, Lemma 1.1 implies that, given $R \in \mathscr{R}, R \in \mathscr{B}$ if and only if $\langle R, S_{ijkl}\rangle = 0$ for all i, j, k, l; i.e., the S_{ijkl} span $\mathscr{S}$ and $\xi \mapsto S_\xi$ maps onto $\mathscr{S}$. Injectivity and the fact that $\xi \mapsto (1/\sqrt{6})S_\xi$ is an isometry follow from taking $R = S_{\alpha\beta\gamma\delta}$ in Lemma 1.1 to conclude that $\{S_{ijkl} \mid i < j < k < l\}$ is an orthogonal set and that $\|S_{ijkl}\|^2 = 6$.

Remark. Using the natural isomorphism between Λ^4 and its dual, the space of alternating 4-forms on V, given by the inner product we can also identify $\mathscr{S}$ with this space of 4-forms. Explicitly, one identifies a 4-form ω on V with the operator $S_\omega \in \mathscr{S}$ given by

$$\langle S_\omega(v_1 \wedge v_2), v_3 \wedge v_4\rangle = \omega(v_1, v_2, v_3, v_4)\,.$$

Proposition 1.3. *Let $\alpha \in \Lambda^2$. Then α is decomposable if and only if $\langle S\alpha, \alpha\rangle = 0$ for all $S \in \mathscr{S}$.*

Proof. The necessity of the condition is clear since each $S \in \mathscr{S}$ is of the form S_ξ for some $\xi \in \Lambda^4$ and $\langle S_\xi\alpha, \alpha\rangle = \langle \alpha \wedge \alpha, \xi\rangle = 0$ for α decomposable.

Conversely. given an orthonormal basis $\{e_1, \cdots, e_n\}$ for V, it is well-known [2, p. 309 ff] (see also [1]) that the conditions $\langle S_{ijkl}\alpha, \alpha\rangle = 0$ for all $i < j < k < l$ are necessary and sufficient conditions for decomposability.

Remark. The conditions $\langle S_{ijkl}\alpha, \alpha\rangle = 0$ are known as the Grassmann quadratic 2-relations.

Remark. It is clear from Proposition 1.3 that each curvature tensor $S \in \mathscr{S}$ has sectional curvature σ_S identically zero. Conversely, it is easily checked that this property characterizes $\mathscr{S}$.

2. The uniqueness theorem

In this section we establish the basic uniqueness result which is at the heart of our building process. But first we need some additional notation.

For a subset Z of G, let

$$\mathscr{A}(Z) = \{S \in \mathscr{S} \mid S(P) = 0 \text{ for all } P \in Z\} .$$

Thus $\mathscr{A}(Z)$ is the subspace of $\mathscr{S}$ consisting of all elements of $\mathscr{S}$ which annihilate Z. For a finite subset $Z = \{P_1, \cdots, P_k\}$ of G, we shall denote $\mathscr{A}(\{P_1, \cdots, P_k\})$ simply by $\mathscr{A}(P_1, \cdots, P_k)$. By $\mathscr{A}(Z)^{\perp}$ with $Z \subset G$ we shall mean the orthogonal complement of $\mathscr{A}(Z)$ in $\mathscr{S}$.

Theorem 2.1. *Let $R \in \mathscr{R}$ and $Z \subset G$, and suppose there exists $S \in \mathscr{S}$ such that $Z \subset \operatorname{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathscr{A}(Z)^{\perp}$ such that $Z \subset \operatorname{Ker}(R - S_0)$. Moreover, given any $S \in \mathscr{S}$, $Z \subset \operatorname{Ker}(R - S)$ if and only if the orthogonal projection of S onto $\mathscr{A}(Z)^{\perp}$ is S_0.*

Proof. Existence: Let $S \in \mathscr{S}$ be such that $Z \subset \operatorname{Ker}(R - S)$, and let S_0 denote the orthogonal projection of S onto $\mathscr{A}(Z)^{\perp}$. Then $S = S_0 + S'$ for some $S' \in \mathscr{A}(Z)$ and

$$Z \subset \operatorname{Ker}(R - S) \cap \operatorname{Ker} S' \subset \operatorname{Ker}(R - S + S') = \operatorname{Ker}(R - S_0).$$

Uniqueness: Suppose $Z \subset \operatorname{Ker}(R - S_0) \cap \operatorname{Ker}(R - S_0')$ for $S_0, S_0' \in \mathscr{A}(Z)^{\perp}$. Then

$$Z \subset \operatorname{Ker}[(R - S_0) - (R - S_0')] = \operatorname{Ker}(S_0' - S_0) .$$

Thus $S_0' - S_0 \in \mathscr{A}(Z)$. But S_0' and $S_0 \in \mathscr{A}(Z)^{\perp}$, so $S_0' - S_0$ must be zero.

Finally, it is immediate from the above existence and uniqueness arguments that $Z \subset \operatorname{Ker}(R - S)$ implies S_0 is the orthogonal projection of S onto $\mathscr{A}(Z)^{\perp}$. Conversely, if $S \in \mathscr{S}$ is such that its orthogonal projection onto $\mathscr{A}(Z)^{\perp}$ is S_0, then $S = S_0 + S'$ for some $S' \in \mathscr{A}(Z)$ and

$$Z \subset \operatorname{Ker}(R - S_0) \cap \operatorname{Ker} S' \subset \operatorname{Ker}(R - S_0 - S') = \operatorname{Ker}(R - S) .$$

Remark. Note that if $R \in \mathscr{R}, S \in \mathscr{S}$ and $P \in G \cap \operatorname{Ker}(R - S)$, then

$$\sigma_R(P) = \langle RP, P\rangle = \langle SP, P\rangle = \sigma_S(P) = 0 .$$

In particular, setting

$$Z(R) = \{P \in G \mid \sigma_R(P) = 0\} ,$$

we see that if, for some $S \in \mathscr{S}$, the subspace $\operatorname{Ker}(R - S)$ has non-null intersection with G then the set $Z(R)$ of zeroes of σ_R is at least big enough to contain this intersection.

Theorem 2.2. *Let $R \in \mathscr{R}$, and suppose there exists $S \in \mathscr{S}$ such that $Z(R) = G \cap \operatorname{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathscr{A}(Z(R))^\perp$ such that $Z(R) = G \cap \operatorname{Ker}(R - S_0)$.*

Proof. By Theorem 2.1, there exists a unique $S_0 \in \mathscr{A}(Z(R))^\perp$ such that $Z(R) \subset G \cap \operatorname{Ker}(R - S_0)$. But, by the remark above, $G \cap \operatorname{Ker}(R - S_0) \subset Z(R)$. Hence we have the equality.

3. Critical zeroes

In studying the critical points of curvature functions, it suffices to consider critical points with critical value zero. For if λ is a critical value of σ_R, $R \in \mathscr{R}$, then the set of critical points of σ_R with critical value λ is the same as the set of critical points of $\sigma_{R-\lambda I}$ with critical value zero, I being the identity operator on Λ^2. In this section we show that if P is a critical zero of σ_R, then $P \in \operatorname{Ker}(R - S)$ for some $S \in \mathscr{S}$.

Lemma 3.1. *Let $P \in G$, and let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V. Then*

$$\{P\} \cup \{S_{ijkl}(P) \mid i < j < k < l\}$$

spans the normal space to $G \subset \Lambda^2$ at P. If the basis is chosen so that $P = e_1 \wedge e_2$, then

$$\{P\} \cup \{S_{12kl}(P) \mid 2 < k < l\}$$

is an orthonormal basis for this normal space.

Proof. By Proposition 1.3,

$$G = \{\alpha \in \Lambda^2 \mid \langle \alpha, \alpha\rangle = 1 \text{ and } \langle S_{ijkl}(\alpha), \alpha\rangle = 0 \text{ for all } i < j < k < l\} .$$

Since the real valued functions $\alpha \mapsto \langle \alpha, \alpha\rangle$ and $\alpha \mapsto \langle S_{ijkl}\alpha, \alpha\rangle$ are constant on G, their gradients $2P$ and $2S_{ijkl}(P)$ at $P \in G$ must be normal to G at P. To see that they span the normal space N_P of G at P, consider first the case where $P = e_1 \wedge e_2$. Then, for $i < j < k < l$,

$$S_{ijkl}(P) = \begin{cases} e_k \wedge e_l , & \text{for } (i, j) = (1, 2) , \\ 0 , & \text{for } (i, j) \neq (1, 2) . \end{cases}$$

It follows that, in this case, $\{P\} \cup \{S_{12kl}(P) \mid 2 < k < l\}$ is an orthonormal set in N_P. Now the number $[(n-2)(n-3)/2]+1$ of elements in this set is equal to the codimension $[n(n-1)/2]-2(n-2)$ of G in Λ^2 which in turn is equal to the dimension of N_P. Hence $\{P\} \cup \{S_{12kl}(P) \mid 2<k<l\}$ is an orthonormal basis for N_P.

Returning to the general case, let $\{e_1, \cdots, e_n\}$ be an arbitrary orthonormal basis for V, and let $\{e'_1, \cdots, e'_n\}$ be one such that $P = e'_1 \wedge e'_2$. Let $\{S_{ijkl} \mid i<j<k<l\}$ and $\{S'_{ijkl} \mid i<j<k<l\}$ be the corresponding bases for $\mathscr{S}$. Then, from above, $\{P\} \cup \{S'_{12kl}(P) \mid 2<k<l\}$ spans N_P. But each S'_{12kl} is a linear combination of the S_{ijkl} and hence each $S'_{12kl}(P)$ is a linear combination of the $S_{ijkl}(P)$. Thus $\{P\} \cup \{S_{ijkl}(P) \mid i<j<k<l\}$ spans N_P.

Theorem 3.2. *Let $R \in \mathscr{R}$ and suppose $P \in G$ is a critical zero of σ_R. Then there exists $S \in \mathscr{S}$ such that $P \in \operatorname{Ker}(R-S)$.*

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V such that $P = e_1 \wedge e_2$. Since P is a critical point of σ_R, and σ_R is the restriction to G of the function $\alpha \mapsto \langle R(\alpha), \alpha \rangle$, the gradient $2R(P)$ of this function at P must be normal to G at P. By Lemma 3.1, this implies that

$$RP = \lambda P + \sum_{2<k<l} \mu_{kl} S_{12kl}(P)$$

for some $\lambda, \mu_{kl} \in \boldsymbol{R}$. But $\lambda = \langle RP, P \rangle = \sigma_R(P) = 0$, so $P \in \operatorname{Ker}(R-S)$ where $S = \sum_{2<k<l} \mu_{kl} S_{12kl}$.

Corollary 3.3. *Let $R \in \mathscr{R}$ and suppose $P \in G$ is a critical zero of σ_R. Then there exists a unique $S \in \mathscr{A}(P)^\perp$ such that $P \in \operatorname{Ker}(R-S)$.*

Proof. Immediate from Theorems 3.2 and 2.1.

Remark. The operator S constructed in the proof of Theorem 3.2 is in fact the unique $S \in \mathscr{A}(P)^\perp$ such that $P \in \operatorname{Ker}(R-S)$. Indeed, by Lemma 1.1 together with the fact that each $S' \in \mathscr{S}$ is an S_ω for some alternating 4-form ω on V, we have

$$\langle S', S_{12kl} \rangle = 6 \langle S'(e_1 \wedge e_2), e_k \wedge e_l \rangle ,$$

and this is zero for all $S' \in \mathscr{A}(P)$; thus $S_{12kl} \in \mathscr{A}(P)^\perp$ for $2<k<l$.

Note also that, since $\{S_{12kl} \mid 2<k<l\}$ is linearly independent, the numbers μ_{kl} above are uniquely determined. In fact, they are curvature components of R relative to the basis $\{e_i\}$:

$$\begin{aligned} \mu_{kl} &= \Big\langle \sum_{2<\alpha<\beta} \mu_{\alpha\beta} e_\alpha \wedge e_\beta, e_k \wedge e_l \Big\rangle = \Big\langle \sum_{2<\alpha<\beta} \mu_{\alpha\beta} S_{12\alpha\beta}(e_1 \wedge e_2), e_k \wedge e_l \Big\rangle \\ &= \langle R(e_1 \wedge e_2), e_k \wedge e_l \rangle . \end{aligned}$$

4. The case $n = 4$

We consider now the case when V has dimension 4, and establish our main theorem in this case. The validity of the result in dimension 4 will play a crucial role in establishing the theorem in general.

Theorem 4.1. *Let* $\dim V = 4$, *and suppose* $R \in \mathscr{R}$ *is such that* $\sigma_R \geq 0$ *and* $Z(R) \neq \emptyset$. *Then there exists a unique* $S \in \mathscr{S}$ *such that* $Z(R) = G \cap \operatorname{Ker}(R - S)$.

Proof. Since $\dim V = 4$, $\mathscr{S}$ is 1-dimensional. Given $\{e_1, \cdots, e_4\}$ an orthonormal basis for V, the operator S_{1234} is just the Hodge star operator $*$ and so $\{*\} = \{S_{1234}\}$ is a basis for $\mathscr{S}$. Given $P \in Z(R)$, P is a minimum, hence a critical point, of σ_R so by Theorem 3.2 there exists $\mu \in \boldsymbol{R}$ such that $P \in \operatorname{Ker}(R - \mu*)$; i.e., such that

$$RP = \mu * P .$$

If P_1 and P_2 are two zeroes of σ_R, then $RP_i = \mu_i * P_i$ for some $\mu_i \in \boldsymbol{R} (i = 1, 2)$. We shall show that $\mu_1 = \mu_2$. This is clear if $\{P_1, P_2\}$ is linearly dependent in Λ^2, so we may assume linear independence. We have

$$\mu_1\langle *P_1, P_2\rangle = \langle RP_1, P_2\rangle = \langle P_1, RP_2\rangle = \mu_2\langle P_1, *P_2\rangle = \mu_2\langle *P_1, P_2\rangle .$$

Hence, if $\langle *P_1, P_2\rangle \neq 0$ we must have $\mu_1 = \mu_2$. On the other hand, if $\langle *P_1, P_2\rangle = 0$, then $\langle P_1 + P_2, *(P_1 + P_2)\rangle = 0$, so $P_1 + P_2$ is decomposable. Let $Q = (P_1 + P_2)/l$ where $l = \|P_1 + P_2\|$. Then $Q \in G$ and

$$RQ = (\mu_1 * P_1 + \mu_2 * P_2)/l ,$$

so $\sigma_R(Q) = \langle RQ, Q\rangle = 0$. Thus Q is also a zero of σ_R; hence $RQ = \mu * Q$ for some $\mu \in R$, and

$$\mu_1 * P_1 + \mu_2 * P_2 = lRQ = l\mu * Q = \mu(*P_1 + *P_2) .$$

This implies that

$$(\mu_1 - \mu)P_1 + (\mu_2 - \mu)P_2 = 0 ,$$

and hence $\mu_1 = \mu_2 = \mu$ since $\{P_1, P_2\}$ is linearly independent in Λ^2.

It follows that $Z(R) \subset \operatorname{Ker}(R - \mu*)$ for some unique $\mu \in \boldsymbol{R}$. By the Remark in §2, $G \cap \operatorname{Ker}(R - \mu*) \subset Z(R)$. Hence, setting $S = \mu*$ we have $Z(R) = G \cap \operatorname{Ker}(R - S)$.

Corollary 4.2. *Let* $\dim V = 4$ *and* $R \in \mathscr{R}$, *and let* λ *denote the minimum (or maximum) value of* σ_R. *Then there exists a unique* $S \in \mathscr{S}$ *such that*

$$\{P \in G \mid \sigma_R(P) = \lambda\} = G \cap \operatorname{Ker}(R - \lambda I - S) .$$

Proof. Follows immediately from Theorem 4.1 upon replacing R in that theorem by $R - \lambda I$ (or, in the case where λ is the maximum value of σ_R, by $\lambda I - R$).

Remark. The hypotheses of Corollary 4.2 cannot by weakened to include the case where λ is an arbitrary critical value of σ_R. Indeed, if we define $R \in \mathscr{R}$ by

$$R(e_1 \wedge e_2) = e_3 \wedge e_4\,, \qquad R(e_3 \wedge e_4) = e_1 \wedge e_2\,,$$
$$R(e_1 \wedge e_3) = 0\,, \qquad R(e_2 \wedge e_4) = 0\,,$$
$$R(e_2 \wedge e_3) = -e_1 \wedge e_4\,, \qquad R(e_1 \wedge e_4) = -e_2 \wedge e_3\,,$$

then each of the basis planes $e_i \wedge e_j$ is a critical zero of σ_R (critical because $(\operatorname{grad} \sigma_R)(e_i \wedge e_j) = 2R(e_i \wedge e_j) = \pm 2{*}e_i \wedge e_j$ which is normal to G at $e_i \wedge e_j$). Hence, if either $\sigma_R^{-1}(0)$ or the critical set of σ_R with critical value zero were the intersection of G with a linear subspace of Λ^2, it would have to be all of G. But this is not the case: setting

$$Q = \tfrac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4 + e_2 \wedge e_3 - e_1 \wedge e_4)$$

we have $Q \in G$ and $\sigma_R(Q) = 1$.

Note that the R of this example satisfies the first Bianchi identity, and also observe that this example illustrates the necessity of the assumption $\sigma_R \geq 0$ (or $\sigma_R \leq 0$) in Theorem 4.1.

Remark. Perhaps a word about the 3-dimensional case is in order at this point, even though it is included in the general case to be considered in the next section. For $n = 3$, every 2-vector is decomposable and hence G is the entire unit sphere in Λ^2. Hence the critical values of σ_R are just the eigenvalues of R, and the set of critical points of σ_R with critical value λ is just the intersection with G of the λ-eigenspace of R. Note that this description (in dimension 3) is valid for each critical value λ, not just for the minimum and maximum values.

5. The main theorem

We now proceed to our main result by way of a sequence of rather technical lemmas.

Lemma 5.1. *Let $R \in \mathscr{R}$ be such that $\sigma_R \geq 0$, and suppose $P, Q \in Z(R)$. Then there exists $S \in \mathscr{S}$ such that $\{P, Q\} \subset \operatorname{Ker}(R - S)$.*

Proof. Choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for V such that $P = e_1 \wedge e_2$ and Q is contained in the span of $\{e_1, \cdots, e_4\}$, so that $Q = \sum_{i<j\leq 4} q_{ij} e_i \wedge e_j$ for some $q_{ij} \in \boldsymbol{R}$. Since Q is a critical point (a minimum) of σ_R, $RQ = \frac{1}{2}(\operatorname{grad} \sigma_R)(Q)$ is normal to G at Q so, by Lemma 3.1,

$$RQ = \sum_{i<j<k<l} \nu_{ijkl} S_{ijkl}(Q) \tag{1}$$

for some $\nu_{ijkl} \in \boldsymbol{R}$ (the component of RQ in the direction of Q is zero since $\langle RQ, Q \rangle = \sigma_R(Q) = 0$). Note that the ν_{ijkl} are not uniquely determined since the $S_{ijkl}(Q)$ are not linearly independent.

Similarly (see the proof of Theorem 3.2),

$$RP = \sum_{2<k<l} \mu_{12kl} S_{12kl}(P)\,, \tag{2}$$

where now the μ_{12kl} are uniquely determined since the $S_{12kl}(P)$ are orthonormal. Moreover, by the Remark following Corollary 3.3, $S_1 = \Sigma \mu_{12kl} S_{12kl}$ is the unique operator in $\mathscr{A}(P)^\perp$ such that $P \in \operatorname{Ker}(R - S_1)$. Thus, by Theorem 2.1, it suffices to construct an $S_2 \in \mathscr{S}$ such that $Q \in \operatorname{Ker}(R - S_2)$ and such that the orthogonal projection of S_2 onto $\mathscr{A}(P)^\perp$ is just S_1. But $\{S_{ijkl} \mid i < j < k < l\}$ is an orthogonal set in $\mathscr{S}$, $S_{12kl} \in \mathscr{A}(P)^\perp$ for $2 < k < l$, and $S_{ijkl} \in \mathscr{A}(P)$ for $(i, j) \neq (1, 2)$, and so the orthogonal projection into $\mathscr{A}(P)^\perp$ of $\sum_{i<j<k<l} \nu_{ijkl} S_{ijkl}$ is just $\sum_{2<k<l} \nu_{12kl} S_{12kl}$. Thus we must show that we can choose $\tilde{\nu}_{ijkl} \in \boldsymbol{R}$ such that

$$RQ = \sum_{i<j<k<l} \tilde{\nu}_{ijkl} S_{ijkl}(Q) \;\; \textit{and} \;\; \tilde{\nu}_{12kl} = \mu_{12kl} \qquad \text{for } 2 < k < l \, .$$

Step I. Given any $\nu_{ijkl}(i < j < k < l)$ such that (1) is satisfied, we shall show that $\nu_{1234} = \mu_{1234}$. Let $W = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \Lambda^4$. Identifying W with the oriented 4-dimensional subspace of V spanned by $\{e_1, \cdots, e_4\}$ we have $P \subset W$ and $Q \subset W$, i.e., $P, Q \in \Lambda^2(W) \subset \Lambda^2(V)$. Letting $\pi_W : \Lambda^2(V) \to \Lambda^2(W)$ denote orthogonal projection, we have

$$\begin{aligned} \nu_{1234} &= \langle \nu_{1234} S_{1234}(Q), S_{1234}(Q) \rangle = \langle \pi_W \Sigma \nu_{ijkl} S_{ijkl}(Q), S_{1234}(Q) \rangle \\ &= \langle \pi_W \circ R(Q), *_W Q \rangle \, , \end{aligned}$$

where $*_W$ is the star operator of W. Similarly,

$$\mu_{1234} = \langle \pi_W \circ R(P), *_W P \rangle \, .$$

But the restriction of $\pi_W \circ R$ to $\Lambda^2(W)$ is a curvature operator (with sectional curvature ≥ 0) on the 4-dimensional space W, and $\{P, Q\}$ is contained in the zero set of this curvature operator. Hence, by Theorem 4.1, there exists a unique $\mu \in \boldsymbol{R}$ such that $P, Q \in \operatorname{Ker}(\pi_W \circ R - S')$ where $S' = \mu *_W$. Thus

$$\nu_{1234} = \langle \pi_W \circ R(Q), *_W Q \rangle = \langle \mu *_W Q, *_W Q \rangle = \mu \, ,$$

and similarly $\mu_{1234} = \mu$, so $\nu_{1234} = \mu_{1234}$.

Step II. We shall take advantage of the non-uniqueness of the remaining ν_{ijkl} in (1) to make essential alterations. In terms of the basis $\{e_i \wedge e_j \mid i < j\}$ for Λ^2, (1) becomes

$$\begin{aligned} RQ = \nu_{1234} S_{1234}(Q) &+ \sum_{5 \leq k} [(\nu_{123k} q_{23} + \nu_{124k} q_{24} + \nu_{134k} q_{34}) e_1 \wedge e_k \\ &\qquad + (-\nu_{123k} q_{13} - \nu_{124k} q_{14} + \nu_{234k} q_{34}) e_2 \wedge e_k \\ &\qquad + (\nu_{123k} q_{12} - \nu_{134k} q_{14} - \nu_{234k} q_{24}) e_3 \wedge e_k \\ &\qquad + (\nu_{124k} q_{12} + \nu_{134k} q_{13} + \nu_{234k} q_{23}) e_4 \wedge e_k] \\ &+ \sum_{5 \leq k < l} [\nu_{12kl} q_{12} + \nu_{13kl} q_{13} + \nu_{14kl} q_{14} \\ &\qquad + \nu_{23kl} q_{23} + \nu_{24kl} q_{24} + \nu_{34kl} q_{34}] e_k \wedge e_l \, . \end{aligned} \tag{3}$$

Case I. Assume $q_{34} \neq 0$. Then, given ν_{ijkl} satisfying (1), we can choose, for each $k \geq 5$, $\tilde{\nu}_{134k}$ and $\tilde{\nu}_{234k} \in \boldsymbol{R}$ so that

$$(4) \qquad \mu_{123k}q_{23} + \mu_{124k}q_{24} + \tilde{\nu}_{134k}q_{34} = \nu_{123k}q_{23} + \nu_{124k}q_{24} + \nu_{134k}q_{34} ,$$

$$(5) \qquad -\mu_{123k}q_{13} - \mu_{124k}q_{14} + \tilde{\nu}_{234k}q_{34} = -\nu_{123k}q_{13} - \nu_{124k}q_{14} + \nu_{234k}q_{34} .$$

(Compare (4) and (5) with the coefficients of $e_1 \wedge e_k$ and $e_2 \wedge e_k$ in (3).) Having chosen $\tilde{\nu}_{134k}$ and $\tilde{\nu}_{234k}$ to satisfy (4) and (5), note that

$$\mu_{123k}q_{12} - \tilde{\nu}_{134k}q_{14} - \tilde{\nu}_{234k}q_{24} = \nu_{123k}(q_{13}q_{24} - q_{14}q_{23})/q_{34}$$
$$-\nu_{134k}q_{14} - \nu_{234k}q_{24} + \mu_{123k}[q_{12} + (q_{14}q_{23} - q_{13}q_{24})/q_{34}] .$$

But

$$q_{12}q_{34} + q_{14}q_{23} - q_{13}q_{24} = \tfrac{1}{2}\langle Q, *_W Q\rangle = 0 ,$$

so the above equation reduces to

$$(6) \qquad \mu_{123k}q_{12} - \tilde{\nu}_{134k}q_{14} - \tilde{\nu}_{234k}q_{24} = \nu_{123k}q_{12} - \nu_{134k}q_{14} - \nu_{234k}q_{24} .$$

(Compare (6) with the coefficient of $e_3 \wedge e_k$ in (3).)
Similarly we can check that

$$(7) \qquad \mu_{124k}q_{12} + \tilde{\nu}_{134k}q_{13} + \tilde{\nu}_{234k}q_{23} = \nu_{124k}q_{12} + \nu_{134k}q_{13} + \nu_{234k}q_{23} .$$

(Compare (7) with the coefficient of $e_4 \wedge e_k$ in (3).)

Finally, since $q_{34} \neq 0$ we can choose, for each $l > k \geq 5$, $\tilde{\nu}_{34kl}$ such that

$$(8) \qquad \mu_{12kl}q_{12} + \tilde{\nu}_{34kl}q_{34} = \nu_{12kl}q_{12} + \nu_{34kl}q_{34} .$$

(Compare (8) with the coefficient of $e_k \wedge e_l$ in (3).)
Then, setting $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$ and $\tilde{\nu}_{ijkl} = \nu_{ijkl}$ for all i, j, k, l for which $\tilde{\nu}_{ijkl}$ has not been previously defined, it follows from (1)–(8), together with step I, that

$$RQ = \Sigma\nu_{ijkl}S_{ijkl}(Q) = \Sigma\tilde{\nu}_{ijkl}S_{ijkl}(Q) ,$$

and $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$. This completes the proof in the case where $q_{34} \neq 0$.

Case II. Suppose $q_{34} = 0$. Then

$$0 = q_{34} = \langle Q, e_3 \wedge e_4\rangle = \langle Q, *_W e_1 \wedge e_2\rangle = \langle Q, *_W P\rangle = \langle P \wedge Q, W\rangle .$$

But $P, Q \in \Lambda^2(W)$ implies $P \wedge Q$ is a multiple of W. Therefore $P \wedge Q = 0$. It follows that the 2-planes P and Q have non-trivial intersection. Hence we can choose our basis $\{e_1, \cdots, e_n\}$ for V so that $P = e_1 \wedge e_2$ and

$$Q = e_1 \wedge (q_{12}e_2 + q_{13}e_3) = q_{12}e_1 \wedge e_2 + q_{13}e_1 \wedge e_3$$

for some $q_{12}, q_{13} \in \boldsymbol{R}$. Since $q_{14} = q_{23} = q_{24} = q_{34} = 0$, (3) becomes

$$(3') \quad \begin{aligned} RQ = \nu_{1234}S_{1234}(Q) &+ \sum_{5\leq k} [\nu_{123k}(-q_{13}e_2 \wedge e_k + q_{12}e_3 \wedge e_k) \\ &\qquad + (\nu_{124k}q_{12} + \nu_{134k}q_{13})e_4 \wedge e_k] \\ &+ \sum_{5\leq k<l} (\nu_{12kl}q_{12} + \nu_{13kl}q_{13})e_k \wedge e_l \,. \end{aligned}$$

Now $\nu_{1234} = \mu_{1234}$ since P and Q both lie in the 4-plane $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ (Step I). Similarly, $\nu_{123k} = \mu_{123k}$ for all $k \geq 4$ since P and Q both lie in the 4-plane $e_1 \wedge e_2 \wedge e_3 \wedge e_k$. Moreover, $q_{13} \neq 0$ since $Q \neq P$, and hence we can choose $\tilde{\nu}_{134k} (k \geq 5)$ and $\tilde{\nu}_{13kl} (l > k \geq 5)$ such that

$$(7') \qquad \mu_{124k}q_{12} + \tilde{\nu}_{134k}q_{13} = \nu_{124k}q_{12} + \nu_{134k}q_{13} \,,$$

$$(8') \qquad \mu_{12kl}q_{12} + \tilde{\nu}_{13kl}q_{13} = \nu_{12kl}q_{12} + \nu_{13kl}q_{13} \,.$$

Then, setting $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$ and $\tilde{\nu}_{ijkl} = \nu_{ijkl}$ for all i, j, k, l for which $\tilde{\nu}_{ijkl}$ has not been previously defined, it follows from (1), (3′), (7′) and (8′) that $RQ = \Sigma \tilde{\nu}_{ijkl} S_{ijkl}(Q)$ and $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$, as required.

Lemma 5.2. *Let $Z \subset G$. Then there exists a finite subset $\{P_1, \cdots, P_k\}$ of Z such that if $R \in \mathscr{R}$ and $P_i \in \operatorname{Ker}(R)$ for all $i \leq k$, then $Z \subset \operatorname{Ker} R$.*

Proof. Suppose not. Then there exists an infinite sequence $\{P_k\}$ in Z such that, for each $k, P_{k+1} \notin \operatorname{Ker}(R)$ for some $R \in \mathscr{R}$ with $\{P_1, \cdots, P_k\} \subset \operatorname{Ker}(R)$. But then

$$\mathscr{R}_k = \{R \in \mathscr{R} \mid \{P_1, \cdots, P_k\} \subset \operatorname{Ker}(R)\}$$

is a strictly decreasing infinite sequence of subspaces of $\mathscr{R}$, contradicting the finite dimensionality of $\mathscr{R}$.

Lemma 5.3. *Let X be an inner product space, and $X_i (1 \leq i \leq k)$ subspaces of X such that $X = \sum_{i=1}^{k} X_i$. Let $\pi_i: X \to X_i$ and $\pi_{ij}: X \to X_i \cap X_j$ $(1 \leq i, j \leq k)$ denote orthogonal projections, and $x_i \in X_i (1 \leq i \leq k)$ be such that $\pi_{ij}x_i = \pi_{ij}x_j$ for all $i \neq j$. Then there exists a unique $x \in X$ such that $\pi_i x = x_i$ for all i.*

Proof. An easy induction on k.

Theorem 5.4. *Let $R \in \mathscr{R}$ be such that $\sigma_R \geq 0$. Then there exists $S \in \mathscr{S}$ such that $Z(R) = G \cap \operatorname{Ker}(R-S)$.*

Proof. We shall construct the unique (see Theorem 2.2) $S \in \mathscr{A}(Z(R))^{\perp}$ which will do the job. By Lemma 5.2, there exists a finite subset $\{P_1, \cdots, P_k\}$ in $Z(R)$ such that every curvature operator which annihilates $\{P_1, \cdots, P_k\}$ annihilates $Z(R)$. In particular,

$$\mathscr{A}(Z(R)) = \mathscr{A}(P_1, \cdots, P_k) = \bigcap_{1 \le i \le k} \mathscr{A}(P_i) \ ,$$

and

$$\mathscr{A}(Z(R))^{\perp} = \sum_{i=1}^{k} \mathscr{A}(P_i)^{\perp} \ .$$

For $i, j \leq k$, let $\pi_i \colon \mathscr{S} \to \mathscr{A}(P_i)^{\perp}$ and $\pi_{ij} \colon \mathscr{S} \to \mathscr{A}(P_i{}^{\perp}) \cap \mathscr{A}(P_j{}^{\perp})$ denote orthogonal projections. By Corollary 3.3, for each $i \leq k$ there exists $S_i \in \mathscr{A}(P_i)^{\perp}$ such that $P_i \in \operatorname{Ker}(R - S_i)$. Moreover, for $i \neq j$, $\pi_{ij}(S_i) = \pi_{ij}(S_j)$. Indeed, by Lemma 5.1, there exists $S_{ij} \in \mathscr{S}$ such that $\{P_i, P_j\} \subset \operatorname{Ker}(R - S_{ij})$ and, by Theorem 2.1, $S_i = \pi_i(S_{ij})$ and $S_j = \pi_j(S_{ij})$ so $\pi_{ij}(S_i) = \pi_{ij}(S_{ij}) = \pi_{ij}(S_j)$. Hence, by Lemma 5.3, there exists $S \in \Sigma \mathscr{A}(P_i)^{\perp} = \mathscr{A}(Z(R))^{\perp}$ such that $\pi_i(S) = S_i$ for all $i \leq k$. By Theorem 2.1 again, this implies that $P_i \in \operatorname{Ker}(R - S)$ for all $i \leq k$, and hence $Z(R) \subset \operatorname{Ker}(R - S)$ by the defining property of the set $\{P_1, \cdots, P_k\}$. Finally, $G \cap \operatorname{Ker}(R - S) \subset Z(R)$ by the remark in § 2 and so we have the equality.

Corollary 5.5. *Let $R \in \mathscr{R}$ and let λ denote the minimum (or maximum) value of σ_R. Then there exists $S \in \mathscr{S}$ such that*

$$\{P \in G \mid \sigma_R(P) = \lambda\} = G \cap \operatorname{Ker}(R - \lambda I - S) \ .$$

Proof. Immediate from Theorem 5.4 upon replacing R in that theorem by $R - \lambda I$ (or, in the maximum case, by $\lambda I - R$).

Remarks. (i) It is interesting to note that the only use of the assumption that λ be the minimum or maximum of σ_R or, in Theorem 5.4, the assumption that $\sigma_R \geq 0$, occurs in the proof of the 4-dimensional case (Theorem 4.1). Thus, if it were true for 4-dimensional spaces that the set of critical points of σ_R with critical value λ were of the form $G \cap \operatorname{Ker}(R - S)$ for some $S \in \mathscr{S}$, then it would be true in general. Of course, it is not. The counterexample in § 4 easily extends to all dimensions ≥ 4.

(ii) Corollary 5.5 implies that there are linear subspaces L_1 and L_2 of Λ^2 such that $G \cap L_1$ is the minimum set of σ_R and $G \cap L_2$ is the maximum set of σ_R. These subspaces can have non-trivial intersection. For example, let $\dim V = 4$ and let $R \in \mathscr{R}$ be defined by

$$\begin{aligned} R(e_1 \wedge e_2) &= R(e_3 \wedge e_4) = e_1 \wedge e_2 + e_3 \wedge e_4 \ , \\ R(e_1 \wedge e_3) &= R(e_2 \wedge e_4) = 0 \ , \\ R(e_1 \wedge e_4) &= R(e_2 \wedge e_3) = -e_1 \wedge e_4 - e_2 \wedge e_3 \ . \end{aligned}$$

Then $L_1 = \operatorname{Ker}(R + I + *)$, $L_2 = \operatorname{Ker}(R - I - *)$, and $\dim(L_1 \cap L_2) = 3$.

(iii) Given any linear subspace L of Λ^2, there exists $R \in \mathscr{R}$ such that $\sigma_R \geq 0$ and $Z(R) = G \cap L$. Indeed, given L, the curvature operator R which is zero on L and identity on $L^{\perp}$ will have these properties. Moreover, the curvature operator obtained by projecting the one just described orthogonally onto $\mathscr{B} = \mathscr{S}^{\perp}$ will have these properties and will in addition satisfy the first Bianchi identity.

(iv) It is a consequence of Corollary 5.5 that if M is an almost Kaehler manifold with almost complex structure J and $m \in M$, then both the set of holomorphic 2-planes at m (planes invariant under J) and the set of anti-holomorphic 2-planes at m (planes P such that $v \in P$ implies $Jv \perp P$) are intersections with G of linear subspaces of $\Lambda^2(V)$ where $V = M_m$ is the tangent space of M at m. Indeed, the automorphism J of V induces a curvature operator, also denoted by J, on V by $J(u \wedge v) = Ju \wedge Jv$ $(u, v \in V)$ and one easily checks that σ_J assumes its maximum value 1 on holomorphic 2-planes and its minimum value 0 on anti-holomorphic 2-planes. A further computation verifies that in fact $P \in G$ is holomorphic if and only if $P \in \operatorname{Ker}(J - I)$, and $P \in G$ is anti-homomorphic if and only if $P \in \operatorname{Ker}(J - S)$ where $S \in \mathscr{S}$ is the operator corresponding under the isomorphisms of § 1 to the 4-form $\varphi \wedge \varphi$, φ being the fundamental 2-form given by $\varphi(u, v) = \langle Ju, v \rangle$.

Added in proof. Theorem 5.4 has recently been generalized by A. Stehney to curvature operators on Λ^p for arbitrary p. Using her techniques, it is possible to eliminate the intricate computations in the proof of Lemma 5.1.

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